

From non-symmetric particle systems to non-linear PDEs on fractals

Joe P. Chen, Michael Hinz, Alexander Teplyaev

Abstract We present new results and challenges in obtaining hydrodynamic limits for non-symmetric (weakly asymmetric) particle systems (exclusion processes on pre-fractal graphs) converging to a non-linear heat equation. We discuss a joint density-current law of large numbers and a corresponding large deviations principle.

Exclusion process on a weighted graph. We consider a locally finite connected (simple and undirected) graph $\Gamma = (V, E)$ with vertex set V and edge set E and endowed with conductances $\mathbf{c} = (c_{xy})_{xy \in E}$ satisfying $c_{xy} > 0$. The pair (Γ, \mathbf{c}) is called a *weighted graph*. Suppose that $H : [0, T] \times V \rightarrow \mathbb{R}$ is a given function with the abbreviated notation $H_t := H(t, \cdot)$. The *weakly asymmetric exclusion process* (WASEP) on (Γ, \mathbf{c}) associated with H is the Markov chain $(\eta_t)_{t \geq 0}$ on $\{0, 1\}^V$ with time-dependent generator $\mathcal{L}_{(\Gamma, \mathbf{c}), H_t}^{\text{EX}}$ defined on functions $f : \{0, 1\}^V \rightarrow \mathbb{R}$ by

$$\left(\mathcal{L}_{(\Gamma, \mathbf{c}), H_t}^{\text{EX}} f \right) (\eta) = \sum_{xy \in E} c_{xy} \psi_{xy}(H_t, \eta) [f(\eta^{xy}) - f(\eta)],$$

where $\psi_{xy}(H_t, \eta) = \exp \{ (\eta(y) - \eta(x)) (H_t(x) - H_t(y)) \}$ and

$$\eta^{xy}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y, \\ \eta(z), & \text{otherwise.} \end{cases}$$

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We can think of $\eta(z)$ as the occupation variable which returns 1 (resp. 0) when z is occupied with a particle (resp. empty). The configuration η^{xy} is obtained by exchanging the occupation variables $\eta(x)$ and $\eta(y)$ in η . At time t such a transition occurs with rate $c_{xy}\psi_{xy}(H_t, \eta)$, where ψ_{xy} encodes the (weak) asymmetry between the hopping rates from x to y and from y to x . When $H \equiv 0$ we obtain the *symmetric exclusion process* (SEP) on (Γ, \mathbf{c}) .

We also define the *boundary-driven* exclusion process. Declare a nonempty subset $\partial V \subset V$ to be the boundary set. Given the aforementioned exclusion process, we add a birth-and-death process to each boundary point $a \in \partial V$; that is, we consider the Markov chain on $\{0, 1\}^V$ generated by

$$\mathcal{L}_{(\Gamma, \mathbf{c}), H_t}^{\text{bEX}} = \mathcal{L}_{(\Gamma, \mathbf{c}), H_t}^{\text{EX}} + \mathcal{L}_{\partial V}^{\text{b}},$$

where for any function $f : \{0, 1\}^V \rightarrow \mathbb{R}$,

$$(\mathcal{L}_{\partial V}^{\text{b}} f)(\eta) = \sum_{a \in \partial V} [\lambda_-(a)\eta(a) + \lambda_+(a)(1 - \eta(a))][f(\eta^a) - f(\eta)],$$

with $\lambda_+(a) > 0$ (resp. $\lambda_-(a) > 0$) representing the birth (resp. death) rate at a , and

$$\eta^a(z) = \begin{cases} 1 - \eta(a), & \text{if } z = a, \\ \eta(z), & \text{otherwise.} \end{cases}$$

We assume that the relative boundary transition rates are bounded away from 0 and ∞ , i.e. we assume there exists $\gamma \in [1, \infty)$ such that $\gamma^{-1} \leq \frac{\lambda_+(a)}{\lambda_-(a)} \leq \gamma$ for all $a \in \partial V$.

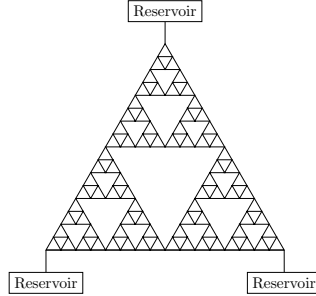


Fig. 1 The pre-Sierpinski gasket graph I_4 , indicating the three reservoirs which fix the particle densities at the boundary points $\{a_0, a_1, a_2\}$.

Analysis on the Sierpinski gasket and exclusion processes on pre-fractal graphs.

We now turn to the Sierpinski gasket (SG). Let a_0, a_1, a_2 be the vertices of an equilateral triangle in \mathbb{R}^2 , and I_0 be the complete graph on the vertex set $V_0 = \{a_0, a_1, a_2\}$. Define the contracting similitude $\Psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\Psi_i(x) = \frac{1}{2}(x - a_i) + a_i$ for each $i \in \{0, 1, 2\}$. For $N \geq 1$, we define $I_N = (V_N, E_N)$ inductively via the formula $I_N = \bigcup_{i=0}^2 \Psi_i(I_{N-1})$. The sequence $(I_N)_{N \geq 0}$ forms a graphical approximation of the

SG fractal K , which is the unique compact set satisfying $K = \bigcup_{i=0}^2 \Psi_i(K)$. By μ we denote the standard self-similar Borel probability measure on K .

A well-known result of Barlow-Perkins [4] states that if for each N we write $(X_t^N)_{t \geq 0}$ to denote the natural symmetric random walk on the approximating graph Γ_N , then the sequence of processes $(X_{5^N}^N)_{N \geq 0}$ is tight and converges to a generic diffusion process (a “Brownian motion”) $(X_t)_{t \geq 0}$ on K . The time acceleration factor 5^N can also be observed in the analytic approach of Kigami [33], in the sense that the sequence of graph energies $\mathcal{E}_N(f) = \frac{5^N}{3^N} \sum_{xy \in E_N} [f(x) - f(y)]^2$, $f : V_N \rightarrow \mathbb{R}$ is monotone and converges to a limit energy form $(\mathcal{E}, \mathcal{F})$ on K . This form is a resistance form in the sense of Kigami, [34], and in particular, it satisfies the Sobolev embedding $\mathcal{F} \subset C(K)$. Moreover, it defines a strongly local regular Dirichlet form on $L^2(K, \mu)$. This allows to define the standard (Dirichlet) Laplacian Δ by a Gauss-Green formula: given $f \in C(K)$, we say that $u \in \text{dom } \Delta$ with $\Delta u = f$ if $\mathcal{E}(u, v) = - \int_K f v d\mu$ for all $v \in \mathcal{F}$ with $v|_{V_0} = 0$. Then $(\Delta, \text{dom } \Delta)$ is a non-positive self-adjoint operator on $L^2(K, \mu)$, and it is just the $L^2(X, \mu)$ -infinitesimal generator of the diffusion $(X_t)_{t \geq 0}$. For more details see [2, 33, 34, 43].

We also need the notions of gradient and divergence. On general weighted graphs they are explained in *e.g.* [40]. We fix an orientation for each edge $xy \in E_N$ of the approximating graph $\Gamma_N = (V_N, E_N)$, and we may assume that the resulting oriented edge, denoted by \vec{xy} , has initial vertex x and terminal vertex y . By $\ell_-^2(E_N)$ we denote the space of functions $\theta : E_N \rightarrow \mathbb{R}$ that are antisymmetric under a change of orientation, *i.e.* satisfy $\theta(\vec{yx}) = -\theta(\vec{xy})$ for all $\vec{xy} \in E$. On each approximating graph Γ_N we can define the *discrete gradient* $\partial_N : \ell^2(V_N) \rightarrow \ell_-^2(E_N)$ by $(\partial_N f)(\vec{xy}) = \frac{5^N}{3^N} [f(y) - f(x)]$. The continuum analogs of ∂_N on the limiting fractal K can be introduced and studied via Dirichlet form theory, *cf.* [12, 19–21, 25–29]. There exist a certain *Hilbert bimodule* \mathcal{H} of *generalized L^2 -vector fields* associated with $(\mathcal{E}, \mathcal{F})$ and a derivation $\partial : \mathcal{F} \rightarrow \mathcal{H}$ that satisfies $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f)$, $f \in \mathcal{F}$. The adjoint $-\partial^* : \mathcal{H} \rightarrow \mathcal{F}$ is then defined in the usual way by duality. In particular,

$$\langle \partial f, g \partial h \rangle_{\mathcal{H}} = -\langle f, \partial^*(g \partial h) \rangle_{L^2(K, \mu)} = \int_K g d\Gamma(f, h)$$

for all $f, g, h \in \mathcal{F}$, where $\Gamma(f, h)$ denote the *mutual energy measure* of f and h , [18, 41, 43]. The operators ∂ and $-\partial^*$ may be viewed as (abstract) *gradient* and *divergence operators* on K . By construction we have $\lim_{N \rightarrow \infty} \|\partial_N f\| = \|\partial f\|_{\mathcal{H}}$ with an appropriate definition of norm involved. However, unlike in the Euclidean setting, we do not have the pointwise convergence $\partial_N f \rightarrow \partial f$.

We consider the boundary-driven WASEP $(\eta_t^N)_{t \in [0, T]}$ on Γ_N generated by the operator $5^N \left(\mathcal{L}_{\Gamma_N, H_t}^{\text{EX}} + \mathcal{L}_{V_0}^b \right)$. Here H is drawn from the space

$$\mathcal{D}_H := \left\{ H : [0, T] \times K \rightarrow \mathbb{R} : \text{for each } x \in K, H_t(x) \text{ is strongly differentiable,} \right. \\ \left. H_t \in \text{dom } \Delta \text{ for all } t \in [0, T], \text{ and } \text{ess sup}_{t \in (0, T)} \mathcal{E}(H_t) < \infty \right\}, \quad (1)$$

and the boundary set V_0 and the rates $\{\lambda_{\pm}(a_i) : i \in \{0, 1, 2\}\}$ are fixed for all N . See Figure 1 for a schematic picture.

We are interested in two observables (empirical measures) in the exclusion process: the *empirical density* and the *empirical integrated current*

$$\pi_t^N(A) = \frac{1}{|V_N|} \sum_{x \in V_N} \eta_t^N(x) \mathbf{1}_A(x), \quad A \subset K,$$

$$\mathbf{W}_t^N(B) = \frac{1}{|E_N|} \sum_{\vec{xy} \in E_N} W_t^N(\vec{xy}) \mathbf{1}_B(\vec{xy}),$$

where B is a subset of oriented edges. Here the *current* $W_t^N(\vec{xy})$, for each oriented edge \vec{xy} in E_N , is the net number of particle jumps along \vec{xy} (i.e. $\#(\text{jumps from } x \text{ to } y) - \#(\text{jumps from } y \text{ to } x)$) in the time interval $[0, t]$. We note the mass conservation law $\sum_{y \in V_N : \vec{xy} \in E_N} W_t^N(\vec{xy}) = -[\eta_t^N(x) - \eta_0^N(x)]$ for $x \in V_N \setminus V_0$.

Hydrodynamic limit. The density-current pair $(\pi_t^N, \mathbf{W}_t^N)$ satisfies a law of large numbers. We fix a macroscopic density ρ_0 satisfying the boundary condition

$$\rho_0(a_i) = \bar{\rho}_i := \frac{\lambda_+(a_i)}{\lambda_+(a_i) + \lambda_-(a_i)}$$

on V_0 , and assume that the sequence of initial densities $(\pi_0^N)_{N \geq 1}$ converges weakly to $\rho_0 d\mu$. We then claim that $(\pi_t^N)_{N \geq 1}$ converges to the weak solution ρ^H of the nonlinear parabolic equation

$$\begin{cases} \partial_t \rho^H(t, x) = \Delta \rho^H(t, x) - \partial^* (\chi(\rho^H(t, x)) \partial H(t, x)) & \text{on } (0, T) \times (K \setminus V_0), \\ \rho^H(0, x) = \rho_0(x), & \text{on } K \setminus V_0, \\ \rho^H(t, a_i) = \bar{\rho}_i & \text{on } (0, T), \end{cases} \quad (2)$$

where $\chi : \mathbb{R} \rightarrow [0, 1]$ is defined by $\chi(s) := (s(1-s))_+$. The quantity $\chi(\rho^H)$ is the mobility of the exclusion process. In addition, the time derivative of \mathbf{W}_t^N converges to the vector field $\mathbf{J}_t = -\partial \rho_t^H + \chi(\rho_t^H) \partial H_t$, which satisfies the macroscopic continuity equation $\partial_t \rho_t^H + \partial^* \mathbf{J}_t = 0$. An important caveat is that these equations are only interpreted in the weak sense, not in the pointwise sense.

Turning to the formal description, we set

$$C_e(K) := \{f \in C(K) : \text{there exists } \varepsilon > 0 \text{ such that } \varepsilon \leq f \leq 1 - \varepsilon\}.$$

Fix $\rho_0 \in C_e(K)$ which satisfies the boundary condition $\rho_0(a_i) = \bar{\rho}_i$. Consider, for every N , the boundary-driven WASEP $(\eta_t^N)_{t \in [0, T]}$ whose initial configurations is η^N , and denote the corresponding law by $\mathbb{P}_{\eta^N}^N$. We assume that $(\eta^N)_{N \geq 0}$ is associated with ρ_0 in the sense of weak convergence, i.e. that

$$\lim_{N \rightarrow \infty} \langle f, \pi_0^N \rangle = \langle f, \rho_0 d\mu \rangle \quad \text{for all } f \in C(K).$$

Here and below $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Now let $\mathcal{F}_0 = \{f \in \mathcal{F} : f|_{V_0} = 0\}$ and \mathcal{F}^* (resp. \mathcal{F}_0^*) be the dual of \mathcal{F} (resp. \mathcal{F}_0). A bounded function $\rho^H \in L^2(0, T, \mathcal{F})$ with $\partial_t \rho^H \in L^2(0, T, \mathcal{F}_0^*)$ is said to be a *weak solution of (2)* if for every $t \in [0, T]$ and every $\varphi \in L^2(0, T, \mathcal{F}_0)$,

$$\int_0^t \int_K (\partial_s \rho_s^H) \varphi_s d\mu ds = - \int_0^t \mathcal{E}(\rho_s^H, \varphi_s) ds + \int_0^t \langle \chi(\rho_s^H) \partial H_s, \partial \varphi_s \rangle_{\mathcal{H}} ds, \quad (3)$$

if $\rho_0^H = \rho_0$ in $L^2(K, \mu)$ and if $\rho_t^H - h \in \mathcal{F}_0$ for a.e. $t \in (0, T)$, where $h \in \mathcal{F}$ is the unique harmonic function on K with boundary values $\bar{\rho}_i$ on V_0 . The specification of the initial condition makes sense, because under the required regularity conditions any such ρ^H will be an element of $C([0, T], L^2(K, \mu))$, see the references mentioned below. There we also comment on the existence and uniqueness of a weak solution ρ^H to (2).

Theorem 1 (Joint density-current large numbers (LLN)). *For every $t \in [0, T]$, $\delta > 0$, $G \in C(K)$, and $F \in \text{dom } \Delta$, we have*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\eta^N}^N \left(\left| \langle \pi_t^N, G \rangle - \int_K G \rho_t^H d\mu \right| > \delta \right) = 0$$

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\eta^N}^N \left(\left| \langle \mathbf{W}_t^N, \partial_N F \rangle - \int_0^t \langle \mathbf{J}_{H_s}(\rho_s^H), \partial F \rangle_{\mathcal{H}} ds \right| > \delta \right) = 0,$$

where ρ^H denotes the unique weak solution of (2), and for a.e. s the vector field $\mathbf{J}_{H_s}(\rho_s^H)$ is defined by the identity

$$\langle \mathbf{J}_{H_s}(\rho_s^H), \partial F \rangle_{\mathcal{H}} := \mathcal{E}(\rho_s^H, F) + \langle \chi(\rho_s^H) \partial H_s, \partial F \rangle_{\mathcal{H}}.$$

We also wish to quantify the probability of the (rare) event that a given trajectory deviates from the hydrodynamic solution (2). This is done by proving a large deviations principle (LDP). Fix $\rho_0 \in C_e(K)$. Let

$$\mathcal{M}_0 := \{\rho(x) d\mu(x) : 0 \leq \rho \leq 1 \text{ } \mu\text{-a.e.}\}$$

be the set of positive measures which are absolutely continuous with respect to μ with density bounded by 1. We will work with $E := D([0, T], \mathcal{M}_0 \times \mathcal{H})$, the space of càdlàg paths from $[0, T]$ to $\mathcal{M}_0 \times \mathcal{H}$ endowed with the Skorokhod topology.

For $H \in \mathcal{D}_H$, we introduce the functional $J_H := J_{H,T,\rho_0} : E \rightarrow \mathbb{R} \cup \{+\infty\}$ as follows. If $(\pi, \mathbf{W}) \notin C([0, T], \mathcal{M}_0 \times \mathcal{H})$, set $J_H(\pi, \mathbf{W}) = +\infty$; otherwise set

$$J_H(\pi, \mathbf{W}) = \langle \partial H_T, \mathbf{W}_T \rangle_{\mathcal{H}} - \int_0^T \left\langle \frac{\partial}{\partial t} \partial H_t, \mathbf{W}_t \right\rangle_{\mathcal{H}} dt - \int_0^T \langle \Delta H_t, \pi_t \rangle dt$$

$$- \int_0^T \langle \chi(\rho_t) \partial H_t, \partial H_t \rangle_{\mathcal{H}} dt + \sum_{i=0}^2 \bar{\rho}_i \int_0^T (\partial^\perp H_t)(a_i) dt.$$

Here $\rho_t = \frac{d\pi_t}{d\mu}$ and ∂^\perp denotes the normal derivative as defined in [33, 43]. Put

$$J(\pi, \mathbf{W}) = \sup_{H \in \mathcal{D}_H} J_H(\pi, \mathbf{W}).$$

Let \mathcal{A} be the set of all $(\pi, \mathbf{W}) \in C([0, T], \mathcal{M}_0 \times \mathcal{H})$ satisfying $d\pi_0 = \rho_0 dv$, $\mathbf{W}_0 = 0$, and the *conservation law* $\partial_t \pi_t + \partial^*(\dot{\mathbf{W}}_t) = 0$ in the weak formulation: for every $\varphi \in \mathcal{F}_0$, $\langle \varphi, \pi_t - \pi_0 \rangle = \langle \partial \varphi, \mathbf{W}_t \rangle_{\mathcal{H}}$. We introduce the *dynamical rate function*

$$I(\pi, \mathbf{W}) = \begin{cases} J(\pi, \mathbf{W}), & \text{if } (\pi, \mathbf{W}) \in \mathcal{A}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4)$$

It turns out that $J(\pi, \mathbf{W})$ can be written in the more symmetric form

$$J(\pi, \mathbf{W}) = \frac{1}{2} \int_0^T \left\langle [\chi(\rho_t)]^{-1} \left(\frac{\partial}{\partial t} \mathbf{W}_t + \partial \rho_t \right), \frac{\partial}{\partial t} \mathbf{W}_t + \partial \rho_t \right\rangle_{\mathcal{H}} dt.$$

Theorem 2 (Joint density-current LDP). *For each closed set \mathcal{C} and each open set \mathcal{O} of E ,*

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{|V_N|} \log \mathbb{P}_{\eta^N}^N ((\pi^N, \mathbf{W}^N) \in \mathcal{C}) &\leq - \inf_{(\pi, \mathbf{W}) \in \mathcal{C}} I(\pi, \mathbf{W}), \\ \liminf_{N \rightarrow \infty} \frac{1}{|V_N|} \log \mathbb{P}_{\eta^N}^N ((\pi^N, \mathbf{W}^N) \in \mathcal{O}) &\geq - \inf_{(\pi, \mathbf{W}) \in \mathcal{O}} I(\pi, \mathbf{W}). \end{aligned}$$

Outline of proof strategy. Our proof strategy is conceptually aligned with the hydrodynamic limit program originating from [23, 36], which has since been expounded in the monograph [35] and applied to various low-dimensional lattice gas models. In particular we are influenced by the works [7, 8, 11]. For an earlier work on the hydrodynamic limit of a related particle system on SG see [30].

That said, we had to overcome a number of technical obstacles to prove the limit theorems on SG . In a nutshell, the difficulties can be attributed to the well-known fact that on SG (and other fractals), the energy measure is singular to the self-similar (Hausdorff) measure [6, 24, 37]. This has two consequences. On the microscopic level, there is no translational invariance, and one needs new tools (resistance-based energy inequalities) to establish a coarse-graining lemma in order to pass to the scaling limit. On the macroscopic level, one needs to utilize notions of vector calculus and (S)PDEs developed through Dirichlet forms.

For the sake of readability we have divided the proofs of Theorems 1 and 2 into several papers, whose contents are summarized as follows:

The moving particle lemma [14]: we prove an energy inequality in the symmetric exclusion process on any finite weighted graph, bounding the cost of swapping particle configurations at vertices x and y in the exclusion process by the effective resistance distance $R_{\text{eff}}(x, y)$ in the random walk process. The proof is based on the *octopus inequality* of [13], which was key to the positive resolution of Aldous' spectral gap conjecture.

A local ergodic (coarse graining) theorem [17]: we show that on many (strongly) recurrent graphs [3, 44, 45], local functions of the occupation variables η in the

(suitably rescaled) exclusion process can be replaced by their macroscopic averages in the scaling limit, with probability superexponentially close to 1. This local ergodic theorem is stated for the conservative process as well as the boundary-driven version. It is proved using the aforementioned moving particle lemma, and can be applied to fractal graphs and trees that lack translational invariance.

Hydrodynamic limit of the exclusion process on SG [15]: we utilize the results of the previous two papers, the vector analysis developed in [26–28], along with the established hydrodynamic limit program, to prove Theorems 1 and 2.

Evolution equations on resistance spaces [16]: we examine the solvability of evolution PDEs which generalize (2) on spaces which support Kigami's resistance forms [34], using tools from Dirichlet forms and the induced vector analysis.

Semilinear hydrodynamic equations. To verify the existence and uniqueness of a weak solution to problem (2), we first consider the problem

$$\begin{cases} \partial_t w(t, x) = \Delta w(t, x) - \partial^* (\chi(w(t, x) + h(x)) \partial H(t, x)) & \text{on } (0, T) \times (K \setminus V_0), \\ w(0, \cdot) = \rho_0 - h, & \text{on } K \setminus V_0, \\ w(\cdot, a_i) = 0 & \text{on } (0, T), \end{cases} \quad (5)$$

Here $h : K \rightarrow \mathbb{R}$ denotes the unique solution $h \in \mathcal{F}$ of the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{on } K \setminus V_0, \\ h(a_i) = \bar{\rho}_i. \end{cases}$$

We translate (5) into an abstract Cauchy problem and solve it using monotone operator methods. The (Dirichlet) Laplacian Δ may be viewed as a bounded variational operator $\Delta : \mathcal{F}_0 \rightarrow \mathcal{F}_0^*$. We consider the nonlinear operator defined by

$$A(t, v) := -\Delta v + \partial^* (\chi(v + h) \partial H_t), \quad v \in \mathcal{F}_0.$$

Recall (1) and in particular, that $H \in L^\infty(0, T, \mathcal{F})$. Together with the resistance form properties of $(\mathcal{E}, \mathcal{F})$ this can be used to see that writing $(\mathcal{A}(u))_t := A(t, u_t)$ for a given function $u : (0, T) \rightarrow \mathcal{F}_0$, we obtain a bounded and demicontinuous operator $\mathcal{A} : L^2(0, T, \mathcal{F}_0) \rightarrow L^2(0, T, \mathcal{F}_0^*)$ which satisfies

$$\lim_{\|u\|_{L^2(0, T, \mathcal{F})} \rightarrow \infty} \frac{\langle \mathcal{A}(u), u \rangle}{\|u\|_{L^2(0, T, \mathcal{F})}} = +\infty. \quad (6)$$

Here we use again $\langle \cdot, \cdot \rangle$ to denote the dual pairing in the obvious sense. We now rephrase (5) as the abstract Cauchy problem

$$\begin{cases} \partial_t w_t + A(t, w_t) = 0 & \text{for a.e. } t \in (0, T) \\ w(0) = \rho_0 - h. \end{cases} \quad (7)$$

A function $w \in L^2(0, T, \mathcal{F}_0)$ with $\partial_t w \in L^2(0, T, \mathcal{F}_0^*)$ is called a (*strong*) *solution* to the abstract Cauchy problem (7) if the first identity holds in \mathcal{F}_0^* (for a.e. $t \in (0, T)$) and the second holds in $L^2(K, \mu)$. Note that any $w \in L^2(0, T, \mathcal{F}_0)$ with $\partial_t w \in$

$L^2(0, T, \mathcal{F}_0^*)$ is a member of $C([0, T], L^2(K, \mu))$, so that the second condition makes sense, see [42, Chapter III, Proposition 1.2].

Proposition 1. *There exists a solution w to the abstract Cauchy problem (7).*

This follows from [39, Théorème 2.1] together with certain estimates based on the resistance form properties of $(\mathcal{E}, \mathcal{F})$. Given a solution w of (7), the function $\rho^H := w + h$ is an element of $L^2(0, T, \mathcal{F})$ and satisfies $\partial_t \rho^H \in L^2(0, T, \mathcal{F}_0^*)$. We also have

$$\partial_t \rho_t^H = \Delta \rho_t^H - \partial^* (\chi(\rho_t^H) \partial H_t) \quad \text{for a.e. } t \in (0, T),$$

seen in \mathcal{F}_0^* , and this implies the validity of the integral identity (3). Moreover, we observe that $\rho_0^H = \rho_0$ and $\rho^H(t, a_i) = \bar{\rho}_i$ for a.e. $t \in (0, T)$.

Theorem 3. *The function ρ^H is the unique weak solution to (2).*

The uniqueness can be obtained by showing that the difference of two solutions has $L^1(K, \mu)$ -norm decreasing in time. This argument was already used in [11, Appendix] and [38, Proof of Lemma 7.1]. It relies on the Markov property of $(\mathcal{E}, \mathcal{F})$. Details are provided in [16].

Perspectives. Theorem 1 may be viewed as a first-principles derivation of a fluid equation on a fractal, subject to external biases on the boundary. The solution of this equation gives the expected trajectory of the macroscopic density and current. Theorem 2 characterizes the fluctuations about this expected trajectory in terms of the large deviations rate function $I(\pi, \mathbf{W})$.

Let us note that on simple graphs which either are discrete tori or have two boundary points (such as $\mathbb{Z} \cap [-n, n]$), there is an alternative combinatorial approach to deriving the rate function [22]. However it is unclear if this approach generalizes to SG (with the standard 3-point boundary) or other infinite weighted graphs. This explains why we followed the hydrodynamic limit program, which has a more robust analytic flavor.

The rate function can be used to analyze properties of macroscopic fluctuations in boundary-driven diffusive processes on networks, which is a current topic of interest in nonequilibrium statistical mechanics; see the excellent recent review [9]. Some highlights in this area include the universality of the cumulant of the mean long-time current in networks with two boundary points [1], under the hypothesis of the *additivity principle* [10]. It is also of interest to investigate the validity of the additivity principle in closely related boundary-driven particle systems [8]. We hope to investigate these and related problems on SG (and other fractals) in the near future, but keep in mind that there are key differences that make the analysis more difficult than on tori and simple lattices. We list a few open related questions:

- Analytically or numerically characterize the infimum of the rate function $J(\pi, \mathbf{W})$ as explicitly as possible.
- Describe the motion of a tagged particle in the exclusion process on SG .
- How does the aforementioned current cumulant problem manifest on SG ?

- Investigate asymmetric exclusion processes and other non-gradient-type particle systems on fractals and other weighted graphs. A key technical step would be to identify energy (spectral gap) inequalities relevant to each process or, in some cases, make use of extra regularity following [5, 31, 32].

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